

LOCAL VOLUME COMPARISON FOR KÄHLER MANIFOLDS

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ABSTRACT. On Kähler manifolds with Ricci curvature lower bound, assuming the real analyticity of the metric, we establish a sharp relative volume comparison theorem for small balls. The model spaces being compared to are complex space forms, i.e, Kähler manifolds with constant holomorphic sectional curvature. Moreover, we give an example showing that on Kähler manifolds, the point-wise Laplacian comparison theorem does not hold when the Ricci curvature is bounded from below.

1. Introduction

Comparison theorems are fundamental tools in geometric analysis. They are vital in the estimates of the spectrums, heat kernels and the Sobolev constants. The classical Bishop-Gromov's relative volume comparison theorem [1][3][4] in Riemannian geometry is the following:

Theorem 1. *Let M^n be a complete Riemannian manifold such that $\text{Ric} \geq (n-1)K$. For any $p \in M$ and $0 < a < b$, the volume of geodesic balls satisfy*

$$\frac{\text{Vol}(B_p(b))}{\text{Vol}(B_p(a))} \leq \frac{\text{Vol}(B_{M_K}(b))}{\text{Vol}(B_{M_K}(a))},$$

where M_K is the simply connected real space form with sectional curvature K , $\text{Vol}(B_{M_K}(r))$ is the volume of the geodesic ball in M_K with radius r . The equality holds iff $B_p(b)$ is isometric to $B_{M_K}(b)$.

The key ingredient in theorem 1 is the Laplacian comparison theorem [2][6]:

Theorem 2. *Let M^n be a complete Riemannian manifold with $\text{Ric} \geq (n-1)K$. Let M_k be the simply connected real space form with sectional curvature K . Denote $r_M(x)$ to be distance function from p to x in M . Let r_{M_k} be the distance function on M_k . Then for any $x \in M$, $y \in M_k$ with $r_M(x) = r_{M_k}(y)$,*

$$\Delta r_M(x) \leq \Delta r_{M_k}(y).$$

The model spaces in above theorems are real space forms. In the Kähler category, it is a natural question whether we can replace the model spaces by Kähler models, i.e, complex space forms which are Kähler manifolds with constant holomorphic sectional curvature. In [5], Li and Wang showed that when the bisectional curvature has a lower bound, both theorems above hold with Kähler models. So the question left is: what can we get if we only assume the lower bound of the Ricci curvature? This note addresses with the local case. The main theorem is the following:

Theorem 3. *Let M^n ($n = \dim_{\mathbb{C}} M$) be a Kähler manifold with real analytic metric. Assume $\text{Ric} \geq K$ (K is any real number). Given any point $p \in M$, there exists $r = r(p, M) > 0$ such that for any $0 < a < b < r$, the volume of geodesic balls satisfy*

$$\frac{\text{Vol}(B_{M^n}(p, b))}{\text{Vol}(B_{M^n}(p, a))} \leq \frac{\text{Vol}(B_{N_K}(b))}{\text{Vol}(B_{N_K}(a))},$$

where N_K denotes the rescaled complex space form with $\text{Ric} = K$, $\Delta_{N_K} r$ is the Laplacian of distance function on N_K . The equality holds iff M is locally isometric to N_K .

Remark 1. *Theorem 3 is a local version of Bishop-Gromov's relative volume comparison theorem on Kähler manifolds. However, one cannot directly extend theorem 3 to any radius. A simple example is the product of \mathbb{P}^1 with the standard product metric. Then the diameter is greater than that of the complex space form. This implies when r is large, the inequality in theorem 3 does not hold.*

We can prove a result which is slightly stronger than theorem 3:

Theorem 4. *Under the same assumption as in theorem 3, there exists $r_0 = r_0(p, M) > 0$ such that for any $r < r_0$, the average Laplacian comparison holds:*

$$\frac{\int_{\partial B_p(r)} \Delta r}{A(\partial B_p(r))} \leq \Delta_{N_K} r(r),$$

where $\Delta_{N_K} r$ is the Laplacian of distance function on N_K . Moreover, the equality holds iff M is locally isometric to N_K .

Remark 2. *Theorem 4 is a local version of theorem 2 in the average sense. However, on Kähler manifolds with Ricci curvature lower bound, the pointwise Laplacian comparison does not hold even locally (see section 6).*

The idea of the proof of theorem 4 is very simple. We shall expand the area of the geodesic sphere $A(\partial B_p(r))$ by power series, then compare the coefficients with that of the rescaled complex space form. The computation is complicated since it involves the covariant derivatives of the curvature tensor with arbitrary order.

This note is organized as follows:

In section 2, we state two propositions which demonstrate the relation between the derivatives of $A(\partial B_p(r))$ and covariant derivatives of the curvature tensor at p . Section 3 is the first part of the proof of proposition 1. We shall estimate the derivatives of $A(\partial B_p(r))$ up to order 4. In the estimate of the 4th derivative, the Kähler condition is employed. The most important part is section 4. We use an induction to prove proposition 1. Besides the routine computation, there are two technical lemmas (lemma 3 and lemma 4) which simplify the computation of higher order covariant derivatives of the curvature tensor significantly. One should note that the Kähler condition is essential in these two lemmas. We complete the proof of proposition 2 and theorem 4 in section 5. The last section is devoted to giving an example showing that the pointwise Laplacian comparing with the complex space form does not necessarily hold if the complex dimension is greater or equal to 2.

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2. Basic set up

Throughout this note, for derivatives of functions of r , we are always evaluating at $r = 0$. Given a point p on a Kähler manifold M^n , fix a unit vector $e_0 \in T_p M$. Along the geodesic l from p with initial direction e_0 , consider the Jacobian equation $J'' = R(e_0, J)e_0$. Set up an orthonormal frame $\{e_k\}$ at p such that $Je_{2i} = e_{2i+1}$, $Je_{2i+1} = -e_{2i}$ for $0 \leq i \leq n-1$. Parallel transport the frame along the geodesic l . Consider the Jacobian field J_u with initial value $J_u(0) = 0$, $J'_u(0) = e_u$.

We may write

$$(2.1) \quad J_u = J_u(r, e_0) = \sum_{i=1}^{\infty} \sum_{v=0}^{2n-1} r^i C_{u,i}^v e_v$$

where $C_{u,i}^v$ are constants independent of r . Denote $R_{e_0 e_u e_0 e_v}$ by R_{uv} when e_0 is fixed. Plugging (2.1) in the Jacobian equation, we get

$$(2.2) \quad \sum_i \sum_v i(i-1) r^{i-2} C_{u,i}^v e_v = \sum_k \sum_w r^k C_{u,k}^w R(e_0, e_w) e_0.$$

Along the geodesic l ,

$$R(e_0, e_w) e_0 = \sum_{s=0}^{2n-1} \sum_{j=0}^{\infty} \frac{R_{sw}^{(j)}}{j!} e_s r^j$$

where $R_{sw}^{(j)}$ denotes the j th covariant derivative of R_{sw} along e_0 at p . Inserting it in (2.2), we get

$$\sum_{i,v} i(i-1) r^{i-2} C_{u,i}^v e_v = \sum_{k,j,w,s} r^{k+j} C_{u,k}^w \frac{R_{sw}^{(j)}}{j!} e_s.$$

Comparing coefficients, we obtain

$$(2.3) \quad C_{u,i}^v = \sum_{k+j=i-2,w} C_{u,k}^w \frac{R_{vw}^{(j)}}{j! i(i-1)}.$$

A simple iteration gives

$$C_{u,1}^v = \delta_u^v; C_{u,2}^w = 0; C_{u,3}^v = \sum_w C_{u,1}^w \frac{R_{vw}}{6} = \frac{R_{uv}}{6};$$

$$C_{u,4}^v = \sum_w C_{u,1}^w \frac{R'_{vw}}{12} = \frac{R'_{vu}}{12};$$

$$C_{u,5}^v = \sum_w (C_{u,1}^w \frac{R''_{vw}}{40} + C_{u,3}^w \frac{R_{vw}}{20}) = \frac{1}{120} (\sum_s R_{us} R_{sv} + 3R''_{uv}).$$

Employing (2.1), we have

$$(2.4) \quad J_u = re_u + \frac{r^3}{6}R_{uv}e_v + \frac{r^4}{12}R'_{uv}e_v + \frac{r^5}{120}(\sum_s R_{us}R_{sv} + 3R''_{uv})e_v + O(r^6).$$

Using dA to denote the standard measure of the unit tangent bundle $UT_p(M)$ at p , via exponential map, we write $\int_{\partial B(p,r)} dA$ as \int . Defining

$$W = \frac{\int \sqrt{\det \langle J_u, J_v \rangle}}{r^{2n-1}},$$

we introduce two propositions as follows:

Proposition 1. *Under the same condition as in theorem 4, if the derivatives of W with order from 1 to $(2m-1)(m \geq 1)$ are the same as that of the complex space form, we have*

Conclusion 1 :

If $m = 2$, $\text{Ric} = K$ at p .

If $m \geq 3$, then $R_{i\bar{j}k\bar{l}} = \frac{K}{n+1}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$ at p . Moreover, for any unit vectors $u, v, e_0 \in UT_p(M)$, $R_{uv}^{(\lambda)} = 0$ for $1 \leq \lambda \leq m-3$ and $\text{Ric}^{(l)}(e_0, e_0) = 0$ for $1 \leq l \leq 2m-4$. The superscripts are orders of covariant derivatives along direction e_0 .

Conclusion 2 : $W^{(2m)}$ is less than or equal to that of the complex space form.

Proposition 2. *Under the same condition as in theorem 4, if the derivatives of W with order from 1 to $(2m)(m \geq 1)$ are the same as the complex space form, $W^{(2m+1)} = 0$.*

We divide the proof of proposition 1 into two parts: $m = 1, 2$ and $m \geq 3$.

3. The proof of proposition 1: Part I

This section treats the case $m = 1, 2$. By (2.1), we have

$$(3.1) \quad \frac{\langle J_u, J_v \rangle}{r^2} = \sum_{i,j,w} r^{i+j-2} C_{u,i}^w C_{v,j}^w.$$

By (2.4),

$$\frac{\langle J_u, J_u \rangle}{r^2} = 1 + \frac{R_{uu}}{3}r^2 + \frac{R'_{uu}}{6}r^3 + \left(\frac{2}{45} \sum_s R_{us}^2 + \frac{1}{20} R''_{uu}\right)r^4 + O(r^5).$$

If $u \neq v$,

$$\frac{\langle J_u, J_v \rangle}{r^2} = \frac{1}{3}R_{uv}r^2 + \frac{R'_{uv}}{6}r^3 + \left(\frac{2}{45} \sum_s R_{us}R_{vs} + \frac{1}{20} R''_{uv}\right)r^4 + O(r^5).$$

Now use the above two expressions to see that

$$(3.2) \quad \begin{aligned} \frac{\det \langle J_u, J_v \rangle}{r^{4n-2}} &= 1 + \frac{1}{3} \sum_u R_{uu}r^2 + \frac{1}{6} \sum_u R'_{uu}r^3 + \left(\frac{2}{45} \sum_{u,s} R_{us}^2 + \frac{1}{20} \sum_u R''_{uu}\right. \\ &\quad \left. + \frac{1}{9} \sum_{u<v} R_{uu}R_{vv} - \frac{1}{9} \sum_{u<v} R_{uv}^2\right)r^4 + O(r^5). \end{aligned}$$

Considering the identity $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$, we get

$$(3.3) \quad \frac{\sqrt{\det \langle J_u, J_v \rangle}}{r^{2n-1}} = 1 + \frac{1}{6} \sum_u R_{uu} r^2 + \frac{1}{12} \sum_u R'_{uu} r^3 + \left(\frac{1}{45} \sum_{u,s} R_{us}^2 + \frac{1}{40} \sum_u R''_{uu} \right. \\ \left. + \frac{1}{18} \sum_{u<v} R_{uu} R_{vv} - \frac{1}{18} \sum_{u<v} R_{uv}^2 - \frac{1}{72} \left(\sum_u R_{uu} \right)^2 \right) r^4 + O(r^5).$$

Since $W = \frac{\int \sqrt{\det \langle J_u, J_v \rangle}}{r^{2n-1}}$, we find

$$W'(0) = 0, W''(0) = -cs$$

where c is a positive constant depending only on n , s is the scalar curvature at p . Therefore $W''(0)$ is less than or equal to that of the complex space form. This proves proposition 1 for $m = 1$.

Now we consider $m = 2$. According to the assumption of proposition 1, W'' is the same as that of the complex space form. Therefore $s = nK$ at p . Since the Ricci curvature is bounded from below by K , $Ric = Kg$ at p . By (3.3), it is simple to see that the r^3 coefficient of W is 0 by symmetry. Thus to complete the proof for $m = 2$, we just need to show that the 4th derivative of W is less than or equal to that of the complex space form.

We keep in mind that $Ric = Kg$ at p . The r^4 coefficient of W is

$$\begin{aligned} c_4 &= \int \left(\frac{1}{45} \sum_{u,s} R_{us}^2 + \frac{1}{40} \sum_u R''_{uu} + \frac{1}{18} \sum_{u<v} R_{uu} R_{vv} - \frac{1}{18} \sum_{u<v} R_{uv}^2 - \frac{1}{72} \left(\sum_u R_{uu} \right)^2 \right) \\ &= \frac{1}{360} \int \left(8 \sum_u R_{uu}^2 + 16 \sum_{u<v} R_{uv}^2 + 9 \sum_u R''_{uu} + 20 \sum_{u<v} R_{uu} R_{vv} \right. \\ &\quad \left. - 20 \sum_{u<v} R_{uv}^2 - 5 \left(\sum_u R_{uu} \right)^2 \right) \\ &= \frac{1}{360} \int \left(-2 \sum_u R_{uu}^2 + 10 \left(\sum_u R_{uu} \right)^2 - 4 \sum_{u<v} R_{uv}^2 + 9 \sum_u R''_{uu} - 5 \left(\sum_u R_{uu} \right)^2 \right) \\ &= \frac{1}{360} \int \left(9 \sum_u R''_{uu} - 4 \sum_{u<v} R_{uv}^2 - 2 \sum_u R_{uu}^2 + 5 \left(\sum_u R_{uu} \right)^2 \right). \end{aligned}$$

Note that the Ricci curvature attains the minimum K at p , so

$$\sum_u R''_{uu} = -Ric''(e_0, e_0) \leq 0.$$

Therefore we have

$$\begin{aligned}
c_4 &= \frac{1}{360} \int (9 \sum_u R''_{uu} - 4 \sum_{u < v} R_{uv}^2 - 2 \sum_u R_{uu}^2 + 5K^2) \\
&\leq -\frac{1}{360} \int (2 \sum_u R_{uu}^2 - 5K^2) \\
&= -\frac{1}{360} \int (2 \sum_{u \neq 1} R_{uu}^2 + 2R_{11}^2 - 5K^2) \\
(3.4) \quad &\leq -\frac{1}{360} \int (\frac{1}{n-1} (\sum_{u \neq 1} R_{uu})^2 + 2R_{11}^2 - 5K^2) \\
&= -\frac{1}{360} \int (\frac{1}{n-1} (Ric(e_0, e_0) + R_{11})^2 + 2R_{11}^2 - 5K^2) \\
&= -\frac{1}{360} \int (\frac{1}{n-1} K^2 + \frac{2}{n-1} KR_{11} + (\frac{1}{n-1} + 2)R_{11}^2 - 5K^2) \\
&\leq -\frac{1}{360} (\int \frac{1}{n-1} K^2 + \frac{2}{n-1} K \int R_{11} + C_1 (\int R_{11})^2 - \int 5K^2) \\
&= C_2 K^2.
\end{aligned}$$

In the inequalities above, C_1, C_2 are constants depending only on n .

We explain the inequalities above. In the first inequality, we drop the two terms $\sum_{u < v} R_{uv}^2$ and $\sum_u R''_{uu}$. In the second inequality, we apply Schwartz inequality for directions e_u that are perpendicular to e_1, e_0 . In the third inequality we use Schwartz inequality $\int R_{11}^2 \geq C(\int R_{11})^2$. We make use of the Kähler condition to obtain $\int R_{11} = C_3 s = nC_3 K$, where C_3 is a constant depending only on n . This explains the last equality.

The right hand side of (3.4) is exactly the case of the complex space form. Therefore when W', W'' are the same as the complex space form, $W^{(3)} = 0$ and $W^{(4)}$ is less than or equal to that of the complex space form. (3.4) becomes an equality if and only if the holomorphic sectional curvature is constant at p and $Ric''(e_0, e_0) = 0$ for any $e_0 \in UT_p M$. This completes the proof for $m = 2$.

4. The proof of proposition 1: Part II

This section deals with the case $m \geq 3$. Denote $Ric^{(l)}(e_0, e_0)$ by $Ric^{(l)}$. According to the assumption of proposition 1, the derivatives of W with order from 1 to $(2m - 1)$ are the same as the complex space form. Follow results in the last section, the holomorphic sectional curvature is constant at p and $Ric'' = 0$ for any e_0 . That is to say, at p ,

$$R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}} = \frac{K}{n+1} (\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}), Ric'' = 0.$$

Therefore, we proved conclusion 1 of proposition 1 for $m = 3$.

Now we use induction. Assuming conclusion 1 of proposition 1 holds for $k = m$, we shall prove that for $k = m + 1$.

Claim 1. *Under the hypothesis of the induction above, $C_{u,i}^v (i \leq m)$ are constants independent of the direction e_0 . In fact, they are the same as that of the complex space form ($C_{u,i}^v$ is defined in (2.1)).*

Proof. Claim 1 follows if we insert the induction hypothesis in (2.3). □

Let us write

$$(4.1) \quad \frac{\det \langle J_u, J_v \rangle}{r^{4n-2}} = 1 + \sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j + O(r^{2m+1}).$$

Combining claim 1 with (3.1), we find that a_i are constants independent of the direction e_0 . (3.1) also yields $C_{u,m+1}^v = C_{v,m+1}^u$ for all u, v . Direct expansion of the determinant via (3.1) gives

$$(4.2) \quad \begin{aligned} b_{2m} = & \sum_{u,v} (C_{u,m+1}^v)^2 + 4 \sum_{u < v} C_{u,m+1}^u C_{v,m+1}^v + 2 \sum_u C_{u,2m+1}^u - 4 \sum_{u < v} C_{u,m+1}^v C_{v,m+1}^u \\ & + \sum_{i=1}^m C_{u,m+i}^v C_{i,m,u,v} + C_{0,m} \end{aligned}$$

where $C_{i,m,u,v}$ and $C_{0,m}$ are all constants independent of the direction e_0 .

Note also

$$(4.3) \quad b_m = 2 \sum_u C_{u,m+1}^u + \text{Constant}.$$

Applying $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \sum_{k=3}^{\infty} \lambda_k x^k (|x| < 1)$, we obtain

$$(4.4) \quad \begin{aligned} \frac{\sqrt{\det \langle J_u, J_v \rangle}}{r^{2n-1}} = & 1 + \frac{1}{2} \left(\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j \right) - \frac{1}{8} \left(\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j \right)^2 \\ & + \sum_{k=3}^{\infty} \lambda_k \left(\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j \right)^k + O(r^{2m+1}). \end{aligned}$$

Lemma 1. *the $2m$ th order coefficient of the expansion of W is*

$$(4.5) \quad \begin{aligned} c_{2m} = & \int \left(\frac{1}{2} \sum_{u,v} (C_{u,m+1}^v)^2 + 2 \sum_{u < v} C_{u,m+1}^u C_{v,m+1}^v + \sum_u C_{u,2m+1}^u \right. \\ & \left. - 2 \sum_{u < v} C_{u,m+1}^v C_{v,m+1}^u - \frac{1}{2} \left(\sum_u C_{u,m+1}^u \right)^2 + \sum_{i=1}^m C_{u,m+i}^v \tilde{C}_{i,m,u,v} \right) + \tilde{C}_{0,m} \end{aligned}$$

where $\tilde{C}_{i,m,u,v}$ and $\tilde{C}_{0,m}$ are constants independent of the direction e_0 .

Proof. It suffices to find out the contribution of each term in (4.4) to c_{2m} . We keep in mind that coefficients a_i in (4.1) are independent of e_0 .

By (4.2), the contribution of term $1 + \frac{1}{2}(\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j)$ to c_{2m} is

$$(4.6) \quad \int \frac{1}{2} \sum_{u,v} (C_{u,m+1}^v)^2 + 2 \sum_{u < v} C_{u,m+1}^u C_{v,m+1}^v + \sum_u C_{u,2m+1}^u - 2 \sum_{u < v} C_{u,m+1}^v C_{v,m+1}^u + \frac{1}{2} \left(\sum_{i=1}^m C_{u,m+i}^v C_{i,m,u,v} + C_{0,m} \right).$$

The contribution of the term $-\frac{1}{8}(\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j)^2$ to c_{2m} is

$$(4.7) \quad - \int \left(\frac{1}{8} b_m^2 + \sum_{i=1}^m C_{u,m+i}^v p_{i,m,u,v} \right) + p_{0,m}.$$

By (4.3), it could be written as

$$(4.8) \quad - \int \left(\frac{1}{2} \left(\sum_u C_{u,m+1}^u \right)^2 + \sum_{i=1}^m C_{u,m+i}^v p_{i,m,u,v} \right) + p_{0,m}.$$

The contribution of $\sum_{k=3}^{\infty} \lambda_k \left(\sum_{i=1}^{m-1} a_i r^i + \sum_{j=m}^{2m} b_j r^j \right)^k$ to c_{2m} is

$$(4.9) \quad \int \sum_{i=1}^m C_{u,m+i}^v q_{i,m,u,v} + q_{0,m}.$$

In (4.7), (4.8), (4.9), $p_{i,m,u,v}$, $q_{i,m,u,v}$, $p_{0,m}$ and $q_{0,m}$ are all constants independent of the direction e_0 . Lemma 1 follows if we combine (4.6), (4.7), (4.8) and (4.9). \square

Lemma 2.

$$(4.10) \quad c_{2m} = \int Q(R_{uv}^{(m-2)}) + \sum_{i=-2}^{m-4} h_{m,i} \int R_{11}^{(m+i)} + C_m \int Ric^{(2m-2)} + Constant$$

where Q is a negative definite quadratic form, $h_{m,i}$ are constants and C_m is a negative constant.

Proof. By the induction hypothesis and (2.3), we have

$$(4.11) \quad \begin{aligned} C_{u,2m+1}^u &= \sum_{k+j=2m-1,w} \frac{C_{u,k}^w R_{uw}^{(j)}}{j!(2m+1)2m} \\ &= \frac{1}{(2m+1)2m} \left(\sum_w \left(\frac{R_{uw}^{(m-2)} C_{u,m+1}^w}{(m-2)!} \right. \right. \\ &\quad \left. \left. + \sum_{j=m-1}^{2m-2} B_{j,m,w,u} R_{uw}^{(j)} + R_{uu} C_{u,2m-1}^u \right) \right) \end{aligned}$$

where $B_{j,m,w,u}$ are constants. For $i \leq m$, we have

$$(4.12) \quad C_{u,m+i}^v = \sum_{j=m-2}^{m+i-3} d_{m,i,j,w,u} R_{uw}^{(j)} + \text{Constant}$$

where $d_{m,i,j,w,u}$ are constants. In particular, we have

$$(4.13) \quad C_{u,m+1}^v = \sum_{k+j=m-1,w} C_{u,k}^w \frac{R_{vw}^{(j)}}{j!m(m+1)} = \frac{1}{m(m+1)} \left(\frac{R_{vu}^{(m-2)}}{(m-2)!} + C_{u,m-1}^v R_{vv} \right).$$

By the induction hypothesis,

$$(4.14) \quad \sum_u R_{uu}^{(m-2)} = -\text{Ric}^{(m-2)} = 0.$$

Therefore

$$(4.15) \quad \begin{aligned} \sum_u (R_{uu}^{(m-2)})^2 &= \left(\sum_u R_{uu}^{(m-2)} \right)^2 - 2 \sum_{u < v} R_{uu}^{(m-2)} R_{vv}^{(m-2)} \\ &= -2 \sum_{u < v} R_{uu}^{(m-2)} R_{vv}^{(m-2)}. \end{aligned}$$

Inserting (4.11), (4.12), (4.13) in (4.5), we find

$$(4.16) \quad c_{2m} = \int Q(R_{uv}^{(m-2)}) + \sum_{i=-2}^{m-2} \int \sum_{u,v} h_{m,i,u,v} R_{uv}^{(m+i)} + \text{Constant}.$$

Now we prove that Q is negative definite. Let us check each term in (4.5).

By (4.13), the term $\frac{1}{2} \sum_{u,v} (C_{u,m+1}^v)^2$ in (4.5) contributes to the quadratic term

$$(4.17) \quad \sum_{u,v} \frac{1}{2m^2(m+1)^2((m-2)!)^2} (R_{uv}^{(m-2)})^2.$$

The term $2 \sum_{u < v} C_{u,m+1}^u C_{v,m+1}^v$ contributes to the quadratic term

$$(4.18) \quad \sum_{u < v} \frac{2}{m^2(m+1)^2((m-2)!)^2} R_{uu}^{(m-2)} R_{vv}^{(m-2)}.$$

By (4.15), it could be written as

$$(4.19) \quad - \frac{1}{m^2(m+1)^2((m-2)!)^2} \sum_u (R_{uu}^{(m-2)})^2.$$

By (4.11) and (4.13), the term $\sum_u C_{u,2m+1}^u$ contributes to the quadratic term

$$(4.20) \quad \sum_{u,v} \frac{1}{2m^2(m+1)(2m+1)((m-2)!)^2} (R_{uv}^{(m-2)})^2.$$

The term $-2 \sum_{u < v} C_{u,m+1}^v C_{v,m+1}^u$ contributes to the quadratic term

$$(4.21) \quad - \sum_{u < v} \frac{2}{m^2(m+1)^2((m-2)!)^2} (R_{uv}^{(m-2)})^2.$$

The term $-\frac{1}{2}(\sum_u C_{u,m+1}^u)^2$ is obvious semi-negative definite.

Combine (4.17), (4.18), (4.19), (4.20) and (4.21), it follows that the quadratic form in (4.10) is negative definite.

Consider the linear terms in (4.16). By the induction hypothesis, the coefficients $h_{m,i,u,v}$ are unchanged if we take a unitary transformation keeping the direction e_0 fixed. Comparing the coefficients of the linear order terms, we see $h_{m,i,u,v} = 0$ if $u \neq v$; $h_{m,i,u,u} = h_{m,i,v,v}$ if $u \neq e_1$ and $v \neq e_1$. Therefore, the linear terms $h_{m,i,u,u}R_{uu}^{(m+i)}$ could be absorbed in $Ric^{(m+i)}$ with the terms $-h_{m,i}R_{11}^{(m+i)}$ left. Also note that by induction hypothesis, $Ric^{(l)} = 0$ for $0 < l \leq 2m-3$ ($Ric^{(2m-3)}$ vanishes as the Ricci curvature attains its minimum at p). Finally, one verifies that $\sum_u C_{u,2m+1}^u$ is the only term in (4.5) that has contribution to $R_{uv}^{(2m-2)}$. Therefore the linear terms in (4.16) could be written as $\sum_{i=-2}^{m-4} h_{m,i} \int R_{11}^{(m+i)} + C_m \int Ric^{(2m-2)}$. From (4.11), it is simple to check that C_m is negative. \square

By the induction hypothesis and that the Ricci curvature attains its minimum at p , we have $Ric^{(2m-2)} \geq 0$. It follows from lemma 2 that

$$(4.22) \quad c_{2m} \leq \sum_{i=-2}^{m-4} h_{m,i} \int R_{11}^{(m+i)} + Constant.$$

We would like to prove that the linear terms $\int R_{11}^{(m+i)}$ vanish for $-2 \leq i \leq m-4$. Note that by symmetry, if $m+i$ is odd, the integral equals 0. Let us deal with case when $m+i$ is even. We shall check when $i = m-4$. Other cases are similar. Let

$$(4.23) \quad A = -\frac{1}{4} \int R_{11}^{(2m-4)}.$$

Set up an orthonormal frame $\{f_i\}$ at p such that $Jf_{2j} = f_{2j+1}$, $Jf_{2j+1} = -f_{2j}$ for $0 \leq j \leq n-1$. Letting $\beta_j = \frac{1}{2}(f_{2j} - \sqrt{-1}f_{2j+1})$, in a small neighborhood of p , we parallel transport the frame along each geodesic through p . Suppose

$$(4.24) \quad e_0 = \sum_{j=0}^{n-1} (z_j \beta_j + \bar{z}_j \bar{\beta}_j).$$

Lemma 3. *Under the assumption of the induction in proposition 1, $Rm^{(\lambda)} = 0$ at p for $1 \leq \lambda \leq m-3$, where $Rm^{(\lambda)}$ denotes any covariant derivative of the curvature tensor with order λ at p .*

Proof. We use induction. If $\lambda = 0$, lemma 3 automatically holds since there is nothing to prove. Suppose lemma 3 holds for $k < \lambda$. For $k = \lambda$, we plug (4.24) in $R_{uv}^{(\lambda)}$.

Claim 2. *We can commute the covariant derivatives for $R_{uv}^{(\lambda)}$.*

Proof. To prove claim 2, we only need to consider the case $\lambda \geq 2$. By the induction hypothesis of lemma 3, the covariant derivatives of the curvature tensor vanish up to order $\lambda - 1$ at p . If $\lambda > 3$, claim 2 follows from the ricci identity. Now suppose $\lambda = 2$. By ricci identity, the difference of commuting the covariant derivatives is a function of the curvature tensor. Note that the curvature tensor at p is the same as the complex space form. We complete the proof for $\lambda = 2$. \square

We insert (4.24) in $R_{Je_0Je_0}^{(\lambda)}$. By claim 2 and Bianchi identities, $R_{Je_0Je_0}^{(\lambda)}$ becomes a polynomial with variables z_j, \bar{z}_j . The coefficients of the polynomial are exactly all the covariant derivatives of Rm at p with order λ . According to the assumption of lemma 3, $R_{Je_0Je_0}^{(\lambda)}$ is identically 0 for all e_0 . Therefore, the coefficients of the polynomial are all 0. This completes the induction of lemma 3. \square

Lemma 4. *Under the assumption of the induction in proposition 1, A could be written as $\sum_{i=1}^{m-2} g_{i,m} \Delta^i s$ where s denotes the scalar curvature, $g_{i,m}$ are constants depending only on n, m, i .*

Proof. Define $X = \frac{1}{2}(e_0 - \sqrt{-1}Je_0)$, then $A = \int R_{X\bar{X}X\bar{X},e_0e_0\dots e_0}$ where the number of e_0 is $2m - 4$. Plugging (4.24) in it, after the integration, we find

$$(4.25) \quad A = \sum_{\alpha_1\alpha_2\dots\alpha_{2m}} \left(\int \alpha_1\alpha_2\dots\alpha_{2m} \right) R_{\alpha_1\alpha_2\alpha_3\alpha_4,\alpha_5\dots\alpha_{2m}}$$

where α_i is $\{z_j\}$ or $\{\bar{z}_k\}$ for $0 \leq j, k \leq n - 1$, $\alpha_1, \alpha_3 \in \{z_j\}$, $\alpha_2, \alpha_4 \in \{\bar{z}_k\}$. Under the subscript of R , z_j stands for β_j , \bar{z}_k stands for $\bar{\beta}_k$.

From the expression of (4.25), we see that z_i, \bar{z}_i must all go in pairs in the sequence $\alpha_1\alpha_2\dots\alpha_{2m}$, otherwise the integral $\int \alpha_1\alpha_2\dots\alpha_{2m}$ equals 0. Switching the covariant derivatives in (4.25), using Kähler identities, we can rearrange (4.25) as

$$(4.26) \quad A = \sum_{I_1, I_2, \dots, I_n} C_{I_1 I_2 \dots I_n} R_{I_1 I_2 \dots I_n} + B$$

where the symbol I_j denotes $z_j \bar{z}_j \dots z_j \bar{z}_j$; subscripts after the fourth subscript of R denote the covariant derivatives; $C_{I_1 I_2 \dots I_n}$ are the coefficients; $\sum_j |I_j| = 2m$; B is the combination of covariant derivatives of Rm with lower order. From (4.23), we see that the coefficients $C_{I_1 I_2 \dots I_n}$ in (4.26) are unitary invariants. For fixed I_3, I_4, \dots, I_n , let $d = |I_1| + |I_2|$. Denote $C_{I_1 I_2 \dots I_n}$ by C_p where $0 \leq |I_1| = p \leq d$. We want to find the relations of $\{C_p\}$. Take a unitary transformation:

$$\tilde{\beta}_i = \beta_i \text{ for } i \neq 1, 2; \beta_1 = \cos \theta \tilde{\beta}_1 + \sin \theta \tilde{\beta}_2; \beta_2 = -\sin \theta \tilde{\beta}_1 + \cos \theta \tilde{\beta}_2.$$

Insert the unitary transformation above in (4.26), the new coefficient \tilde{C}_d becomes

$$\sum_{p=0}^d C_p \cos^{2p} \theta \sin^{2(d-p)} \theta. \text{ Therefore we have:}$$

$$(4.27) \quad \sum_{p=0}^d C_p \cos^{2p} \theta \sin^{2(d-p)} \theta = C_d = C_d (\cos^2 \theta + \sin^2 \theta)^d.$$

Claim 3. $C_p = C_d \binom{d}{p}$

Proof. Divide by $\cos^{2d}\theta$ on both sides, (4.27) becomes

$$\sum_{p=0}^d C_p \tan^{2(d-p)} \theta = C_d = C_d(1 + \tan^2 \theta)^d.$$

Since θ is arbitrary, claim 3 follows. \square

By claim 3, $\frac{C_p}{C_d} = \binom{d}{p}$. Since we can substitute any index u, v for 1, 2, the ratio of all coefficients in (4.26) are determined. Note that to get the relations between C_p , we only use the condition that the form (4.23) is unitary invariant. Since $\Delta^{m-2}s$ is also unitary invariant with respect to the frame, we can write it in the form as (4.26). By the same argument, the ratio of all coefficients of $\Delta^{m-2}s$ are the same as (4.26). It follows that the term $\sum_{I_1, I_2, \dots, I_n} C_{I_1 I_2 \dots I_n} R_{I_1 I_2 \dots I_n}$ in (4.26) equals $C(m, n)\Delta^{(m-2)}s$ modulo lower order covariant derivatives, where $C(m, n)$ is a constant depending only on m, n .

Now we make an important observation. From the Ricci identity, $R_{i_1 i_2 \dots i_p \alpha \beta i_{p+3} \dots i_{2m}} - R_{i_1 i_2 \dots i_p \beta \alpha i_{p+3} \dots i_{2m}}$ is the sum of $(Rm Rm^{(p-4)})_{i_{p+3} \dots i_{2m}}$. By lemma 3, $Rm^{(\lambda)} = 0$ for $1 \leq \lambda \leq m-3$. It follows that $(Rm Rm_{i_5 \dots i_p})_{i_{p+3} \dots i_{2m}}$ can be expanded as a linear combination of the covariant derivatives of curvature tensor. Therefore $A - C(m, n)\Delta^{(m-2)}s$ can be written as a linear combination of the covariant derivatives of the curvature tensor with the highest order $2m - 6$. Furthermore it is unitary invariant since the curvature tensor is unitary invariant at p . By recursive arguments, we complete the proof of lemma 4. \square

From the induction in proposition 1, $Ric^{(l)} = 0$ for $1 \leq l \leq 2m - 4$. Integrating with respect to the unit sphere in $T_p M$, by similar arguments as in the proof of lemma 4, we find that for l even,

$$(4.28) \quad 0 = \int Ric_{e_0 e_0, e_0 e_0 \dots e_0} = \sum_{k=1}^{\frac{l}{2}} C_{l,k} \Delta^k s$$

where the order of the covariant derivative above is l . It is straightforward to check that the highest order coefficient $C_{l, \frac{l}{2}}$ is not equal to 0. Then by a recursive argument, $\Delta^k s = 0$ at p for $1 \leq k \leq m - 2$. Combine this with lemma 4, it follows that $A = 0$. Similarly all linear terms in (4.10) vanish. Therefore, under the induction hypothesis in proposition 1, in order that c_{2m} in (4.10) achieves the maximum, $Ric^{(2m-2)} = 0$ and $R_{uv}^{(\lambda)} = 0$ for $1 \leq \lambda \leq m - 2$. This is exactly the case of the complex space form. Therefore we complete the induction in proposition 1. As a byproduct, we proved conclusion 2 in proposition 1. The proof of proposition 1 is complete. \square

5. The proof of theorem 4

Under the assumption of proposition 2, using the same argument as in the last section, we find that $W^{(2m+1)}$ is a linear combination of $\int R_{11}^{(m+i)} (1 \leq i \leq m-3)$ (the

terms with order greater than $2m - 3$ could be absorbed in $Ric^{(m+i)}$ to vanish). Similar as the proof of lemma 4, $W^{(2m+1)}$ is equal to 0. This completes the proof of proposition 2.

Consider two cases below:

1. All coefficients of the power series of W are equal to that of the complex space form. Follow proposition 1, all covariant derivatives of the curvature tensor at p are the same as the complex space form. Since the metric is real analytic, we conclude that near p , the manifold is isometric to the complex space form.

2. There is a $i_0 \geq 1$ such that for all $i < i_0$, the coefficients of the power series of W are equal to that of the complex space form, but the i_0 th coefficient is less than that of the complex space form. Checking the power series of $\frac{W'}{W}$ at p , we find that for sufficiently small r , $\frac{W'}{W}$ is less than that of the complex space form. Follow the definition of W , for small r ,

$$\frac{\int_{\partial B_p(r)} \Delta r}{A(\partial B_p(r))} = \frac{\int (\sqrt{\det \langle J_u, J_v \rangle})'}{\int \sqrt{\det \langle J_u, J_v \rangle}} < \Delta_{N_K} r(r).$$

The proof of theorem 4 is complete. \square

6. An example

In this section we give an example showing that the analogous Laplacian comparison theorem is not true on Kähler manifolds when the Ricci curvature is bounded from below by a nonzero constant. The example is in dimension 2. For higher dimensions, the construction is similar.

Identify \mathbb{R}^4 with \mathbb{C}^2 in the usual way. The corresponding almost complex structure J is given by $J \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2}$, $J \frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x_1}$, $J \frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4}$, $J \frac{\partial}{\partial x_4} = -\frac{\partial}{\partial x_3}$.

Given a small ball near the origin of \mathbb{C}^2 , define the function f to be

$$f = |z_1|^2 + |z_2|^2 + a|z_1|^4 + 8a|z_1|^2|z_2|^2 + a|z_2|^4 + \frac{8}{3}a^2|z_1|^6 + 28a^2|z_1|^4|z_2|^2 + 28a^2|z_1|^2|z_2|^4 + \frac{8}{3}a^2|z_2|^6 + p(|z_1|, |z_2|)$$

where a is a nonzero constant and p is a homogeneous polynomial of degree 8 which will be determined later.

We define

$$\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} f = \frac{\sqrt{-1}}{2} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j.$$

It is straightforward to check that ω defines a Kähler metric g if the ball is sufficiently small (note that the metric is not complete).

Direct computation gives

$$g_{1\bar{1}} = 1 + 4a|z_1|^2 + 8a|z_2|^2 + 24a^2|z_1|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_2|^4 + O((|z_1| + |z_2|)^6);$$

$$\begin{aligned}
g_{2\bar{2}} &= 1 + 4a|z_2|^2 + 8a|z_1|^2 + 24a^2|z_2|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_1|^4 \\
&\quad + O((|z_1| + |z_2|)^6)); \\
g_{1\bar{2}} &= 8a\bar{z}_1z_2 + 56a^2z_1\bar{z}_1^2z_2 + 56a^2\bar{z}_1z_2^2\bar{z}_2 + O((|z_1| + |z_2|)^6)).
\end{aligned}$$

Therefore

$$\begin{aligned}
\det(g_{i\bar{j}}) &= g_{1\bar{1}}g_{2\bar{2}} - |g_{1\bar{2}}|^2 \\
&= (1 + 4a|z_1|^2 + 8a|z_2|^2 + 24a^2|z_1|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_2|^4) \\
&\quad (1 + 4a|z_2|^2 + 8a|z_1|^2 + 24a^2|z_2|^4 + 112a^2|z_1|^2|z_2|^2 + 28a^2|z_1|^4) \\
&\quad - |8a\bar{z}_1z_2 + 56a^2z_1\bar{z}_1^2z_2 + 56a^2\bar{z}_1z_2^2\bar{z}_2|^2 + O((|z_1| + |z_2|)^6) \\
&= 1 + 12a(|z_1|^2 + |z_2|^2) + 84a^2(|z_1|^4 + |z_2|^4) + 240a^2|z_1|^2|z_2|^2 \\
&\quad + O((|z_1| + |z_2|)^6)).
\end{aligned}$$

Using $\log(1+x) = x - \frac{1}{2}x^2 + O(x^3)$, we have

$$Ric + 12ag = \partial\bar{\partial}(-\log(\det g_{i\bar{j}}) + 12af) = \partial\bar{\partial}(O((|z_1| + |z_2|)^6)).$$

Therefore $Ric + 12ag$ vanishes up to order 3 at the origin. Moreover, if we choose the function p to be $-\lambda(|z_1|^8 + |z_2|^8 + 8(|z_1|^6|z_2|^2 + |z_1|^2|z_2|^6))$, after a direct computation,

$$Ric + 12ag = \partial\bar{\partial}(24\lambda(|z_1|^2 + |z_2|^2)^3 + O((|z_1| + |z_2|)^6))$$

where the term $O((|z_1| + |z_2|)^6)$ does not depend on λ . If λ is sufficiently large, $Ric + 12ag \geq 0$ near the origin. Set $K = -12a$. Thus, near the origin, $Ric \geq K$. By direct computation, at the origin, $R_{1212} = R_{1313} = R_{1414} = 4a$; $R_{1u1v} = 0$ if $u \neq v$. Combining this with the fact that the second derivatives of the Ricci tensor vanish at the origin, after a slight computation, we find that the fourth order term of (3.3) is greater than that of the complex space form if $e_0 = \frac{\partial}{\partial x_1}$. So when r is very small, along the geodesic with initial direction $\frac{\partial}{\partial x_1}$ at the origin, $\sqrt{\det \langle J_u, J_v \rangle}$ is greater than that of the complex space form. Since $\Delta r = \frac{\partial \log \sqrt{\det \langle J_u, J_v \rangle}}{\partial r}$, it follows that the pointwise Laplacian comparing with the complex space forms is not true for Kähler manifolds.

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